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MINIMAL IMBEDDINGS OF *R*-SPACES

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1. Introduction

Let G be a connected real semi-simple Lie group without center and U a parabolic subgroup of G. The quotient space G/U is called an *R-space*. A maximal compact subgroup K of G is transitive on G/U so that an R-space is necessarily compact. Let $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ be a Cartan decomposition of the Lie algebra $\mathfrak{G}$ of G with respect to the Lie algebra $\mathfrak{K}$ of K. The main purpose of this paper is to construct a natural imbedding φ of an R-space G/U into $\mathfrak{P}$ with the following properties:

(1) φ is K-equivariant;

(2) φ has minimum total curvature;

(3) If G is simple and $K/K \cap U$ is a symmetric space, then φ is isometric and $\varphi(G/U)$ is a minimal submanifold of a hypersphere in $\mathfrak{P}$ in the sense that its mean curvature normal is zero.

In general, an n-dimensional submanifold M of the hypersphere $S^N(r)$ of radius r about the origin in the Euclidean space $\boldsymbol{R}^{N+1}$ is a minimal submanifold if and only if

$$\Delta y^i = -\frac{n}{r^2} y^i \quad \text{on } M \text{ for } i = 1, \cdots, N+1 ,$$

where $(y^1, \cdots, y^{N+1})$ is a coordinate system for $\boldsymbol{R}^{N+1}$ and Δ is the Laplacian of M. For many symmetric R-spaces we verify that the Laplacian Δ for functions has no eigen-value between 0 and $-n/r^2$. We do not know whether this is true or not in general for all symmetric R-spaces.

Previously, it was known that φ has minimum total curvature if G/U is a Käehlerian C-space (Kobayashi [6]) or if G/U is a symmetric space of rank 1 (Tai [15]). For a symmetric R-space G/U, the imbedding φ has been considered by Nagano [13], and has also been conjectured to have minimum total curvature (Kobayashi [7]). The class of symmetric R-spaces includes

(i) all hermitian symmetric spaces of compact type;

(ii) Grassmann manifolds $O(p+q)/O(p) \times O(q)$, $Sp(p+q)/Sp(p) \times Sp(q)$;

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(iii) the classical groups $SO(m)$, $U(m)$, $Sp(m)$;

(iv) $U(2m)/Sp(m)$, $U(m)/O(m)$;

(v) $(SO(p+1) \times SO(q+1))/S(O(p) \times O(q))$, where $S(O(p) \times O(q))$ is the subgroup of $SO(p+1) \times SO(q+1)$ consisting of matrices of the form

$$\begin{pmatrix} \varepsilon & 0 & & \\ 0 & A & & \\ & & \varepsilon & 0 \\ & & 0 & B \end{pmatrix}, \quad \varepsilon = \pm 1 , \quad A \in O(p) , \quad B \in O(q) ;$$

(This R-space is covered twice by $S^p \times S^q$.)

(vi) the Cayley projective plane and three exceptional spaces.

An explicit formula for the imbedding φ of a symmetric R-space of classical type in $\mathfrak{P}$ in terms of matrices can be found in Kobayashi [7].

In § 3 we recall briefly the concept of minimum imbedding without mentioning that of total curvature. For the latter we refer the reader to Chern and Lashof [1], [2], Kuiper [9], [10] and references therein.

The result of this paper on the total curvature of φ relies heavily on the cellular decomposition of an R-space obtained by Takeuchi [16].

Our result on minimal submanifolds of a hypersphere is somewhat related to those of Takahashi [7] and Hsiang [4], and Proposition 5.1 on minimal submanifolds appears in Takahashi [17].

2. Parabolic subgroups and R-spaces

Let G be a connected real semi-simple Lie group without center, and $\mathfrak{G}$ its Lie algebra. Let $\mathfrak{G}_C$ be the complexification of $\mathfrak{G}$, and G_C the connected complex semi-simple Lie group without center generated by the Lie algebra $\mathfrak{G}_C$. Then we may consider G as a subgroup of G_C. The complex conjugation σ of $\mathfrak{G}_C$ with respect to $\mathfrak{G}$ generates an automorphism σ of G_C which leaves G elementwise fixed.

A subgroup of G_C is called a *parabolic subgroup* of G_C if it contains a maximal solvable subgroup of G_C; it is always connected. A subgroup of G is called a *parabolic subgroup* of G if it is the intersection of G and a σ-invariant parabolic subgroup of G_C. A parabolic subgroup of G may not be connected, but it is still uniquely determined by its Lie algebra alone. A subalgebra of $\mathfrak{G}$ is called a *parabolic subalgebra* if it is the Lie algebra of a parabolic subgroup of G. If Z is an element of $\mathfrak{G}$ such that $ad\, Z$ is a semi-simple endomorphism of $\mathfrak{G}$ whose eigen-values are all real, then the direct sum $\mathfrak{U}$ of all eigen-spaces corresponding to the non-negative eigen-values of $ad\, Z$ is a parabolic subalgebra of $\mathfrak{G}$. Conversely, every parabolic subalgebra of $\mathfrak{G}$ can be obtained in this fashion (cf. Matsumoto [11]).

An *R-space* is, by definition, a quotient space $M = G/U$, where G is a connected real semi-simple Lie group without center and U is a parabolic subgroup of G. Given an R-space $M = G/U$, we choose once and for all an element $Z \in \mathfrak{G}$ which determines the parabolic subalgebra $\mathfrak{U}$, the Lie algebra of U, in the manner described above. (Such an element Z is not unique.) We choose also a maximal compact subgroup K of G such that Z is perpendicular to the Lie algebra $\mathfrak{K}$ of K with respect to the Killing form $(\, , \,)$ of $\mathfrak{G}$. In the Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$, Z is then contained in $\mathfrak{P}$. We choose a maximal abelian subalgebra $\mathfrak{A}$ of $\mathfrak{P}$, which contains Z, and introduce a linear order in the dual space of $\mathfrak{A}$ in such a way that $\gamma(Z) \geq 0$ for all positive roots γ of $\mathfrak{G}$ with respect to $\mathfrak{A}$. Let $\mathfrak{N}$ be the direct sum of the root spaces corresponding to the positive roots. Then $\mathfrak{N}$ is a nilpotent subalgebra of $\mathfrak{G}$. Let N be the connected subgroup of G generated by $\mathfrak{N}$, and set

$$K_0 = \{k \in K;\ (Ad\,k)Z = Z\}\ .$$

Then we have (Takeuchi [16])

Proposition 2.1. (i) *$KU = G$ and $K \cap U = K_0$ so that $M = K/K_0$;* (ii) *If we denote by $N_K(\mathfrak{A})$ (resp. $N_{K_0}(\mathfrak{A})$) the normalizer of $\mathfrak{A}$ in K (resp. in K_0), then $N_K(\mathfrak{A})/N_{K_0}(\mathfrak{A})$ is finite. If $k_1, \cdots, k_b \in N_K(\mathfrak{A})$ are complete representatives of $N_K(\mathfrak{A})/N_{K_0}(\mathfrak{A})$ and if o denotes the origin of G/U, then the orbits $Nk_1o, \cdots, Nk_bo$ of N through $k_1o, \cdots, k_bo$ give a cellular decomposition of M, and these cells are all cycles mod 2.*

As a consequence, we have $\sum_i \dim H_i(M, Z_2) = b$. From (i) we see that the mapping $\varphi : M = K/K_0 \to \mathfrak{P}$ defined by

$$\varphi(kK_0) = (Ad\,k)Z\ , \qquad kK_0 \in K/K_0$$

is a K-equivariant imbedding of M into $\mathfrak{P}$. The purpose of this paper is to study geometric properties of this imbedding φ.

Proposition 2.2. *Let X be a regular element of $\mathfrak{P}$. Then the number of zero points of the vector field on M generated by X coincides with the number b of the elements in $N_K(\mathfrak{A})/N_{K_0}(\mathfrak{A})$.*

Proof. We first prove

Lemma. *If we set $\mathfrak{P}_0 = \{X \in \mathfrak{P};\ [Z, X] = 0\}$, then $\mathfrak{U} \cap \mathfrak{P} = \mathfrak{P}_0$.*

Proof of Lemma. From the definitions of $\mathfrak{U}$ and $\mathfrak{P}_0$ we have clearly $\mathfrak{P}_0 \subset \mathfrak{U} \cap \mathfrak{P}$. Let $X \in \mathfrak{U} \cap \mathfrak{P}$ and write

$$X = X_0 + X_+\ ,$$

where $[Z, X_0] = 0$ and X_+ is in the direct sum of the eingen-spaces corresponding to the positive eigen-values of $ad\,Z$. We wish to show $X_+ = 0$. Let τ be the involutive automorphism of $\mathfrak{G}$ such that $\tau|_{\mathfrak{K}} =$ identity and $\tau|_{\mathfrak{P}} =$ $-$identity. Then $\tau Z = -Z$ and hence $\tau \circ (ad\,Z) = -(ad\,Z) \circ \tau$. It follows that $[Z, \tau X_0] = 0$ and that τX_+ is in the direct sum of the eigen-spaces corresponding to the negative eigen-values of $ad\,Z$. On the other hand, since X

is in $\mathfrak{P}$, we have $\tau X = -X$ and $\tau X \in \mathfrak{U} \cap \mathfrak{P}$. Since $\tau X = \tau X_0 + \tau X_+$ is in $\mathfrak{U}$, it follows that $X_+ = 0$. This completes the proof of the lemma.

Let X be a regular element of $\mathfrak{P}$. For each $k \in K$, X and $(Ad\, k)X$ generate vector fields on M with the same number of zero points on M. Since $(Ad\, k)X \in \mathfrak{A}$ for a suitable k, we may assume that X is a regular element of $\mathfrak{A}$. It suffices therefore to prove that, for a regular element X of $\mathfrak{A}$, the zero points of the vector field generated by X coincide with the orbit $N_K(\mathfrak{A})o$ of $N_K(\mathfrak{A})$ through the origin o of $M = K/K_0$. Let ko $(k \in K)$ be a zero point of the vector field generated by X. Then $X \in (Ad\, k)\mathfrak{U}$ and hence $(Ad\, k^{-1})X \in \mathfrak{U}$. Since $(Ad\, k^{-1})X \in \mathfrak{P}$, the lemma above implies $(Ad\, k^{-1})X \in \mathfrak{P}_0$. If we set $\mathfrak{G}_0 = \{Y \in \mathfrak{G};\ [Z, Y] = 0\}$, then $\mathfrak{G}_0$ is a reductive Lie algebra, and $\mathfrak{G}_0 = \mathfrak{K}_0 + \mathfrak{P}_0$ is a Cartan decomposition of $\mathfrak{G}_0$. Since $\mathfrak{A}$ is a maximal abelian subalgebra of $\mathfrak{P}_0$, there exists an element $k_0 \in K_0$ such that $(Ad\, k_0^{-1})(Ad\, k^{-1})X \in \mathfrak{A}$. If we set $k' = kk_0$, then $(Ad\, k'^{-1})X \in \mathfrak{A}$. Since X is a regular element of $\mathfrak{A}$, k' lies in $N_K(\mathfrak{A})$. On the other hand, $k'o = kk_0o = ko$. It is easy to see the converse that $N_K(\mathfrak{A})o$ is contained in the set of zero points of the vector field generated by X.

3. Minimum imbeddings

Let M be a compact manifold, and $\mathcal{F}$ the set of C^∞ functions f on M whose critical points are all isolated and non-degenerate. For each $f \in \mathcal{F}$, we denote by $\beta(f)$ the number of the critical points of f on M. Set

$$\beta = \inf_{f \in \mathcal{F}} \beta(f) .$$

Then β depends only on the differentiable structure of M, and the theory of Morse tells us that, for any coefficient field F, the following inequality holds:

$$\beta \geq \textstyle\sum_i \dim H_i(M, F) .$$

Let φ be an imbedding of M into a real vector space V. Then for almost[1] all linear functional u on V, the function $u \circ \varphi$ belongs to the family $\mathcal{F}$. We say that the imbedding $\varphi: M \to V$ is *minimum* if $\beta = \beta(u \circ \varphi)$ for almost all linear functionals u on V such that $u \circ \varphi$ belongs to the family $\mathcal{F}$. Since $\beta(u \circ \varphi) \geq \beta \geq \sum_i \dim H_i(M, F)$ always, φ is minimum if $\beta(u \circ \varphi) = \sum_i \dim H_i(M, F)$ for some coefficient field F and almost all linear functionals u such that $(u \circ \varphi) \in \mathcal{F}$.

We shall prove the following theorem:

Theorem 3.1. *Let $M = G/U$ be an R-space, and $\varphi: M \to \mathfrak{P}$ the imbedding defined in* § 2. *Then φ is minimum, and*

[1] in the sense of measure.

$$\beta = \sum_i H_i(M, Z_2) .$$

We shall first outline the proof. Let X be any element of $\mathfrak{P}$, and u_X the linear functional on $\mathfrak{P}$ which corresponds to X under the duality defined by the Killing form (,) of $\mathfrak{G}$. We define a suitable Riemannian metric $\ll , \gg$ and show that the 1-form $d(u_X \circ \varphi)$ corresponds to the vector field generated by X by the duality defined by $\ll , \gg$. Then the critical points of $u_X \circ \varphi$ coincide with the zero points of the vector field generated by X. Since the singular elements of $\mathfrak{P}$ form a set of measure zero, the theorem will then follow immediately from Propositions 2.1 and 2.2. We now give the details of the proof.

Let $\mathfrak{K}_0$ be the Lie algebra of K_0. The Killing form (,) of $\mathfrak{G}$ is negative definite on $\mathfrak{K}$. Let $\mathfrak{M}$ be the orthogonal complement of $\mathfrak{K}_0$ in $\mathfrak{K}$ with respect to the Killing form (,). Then $\mathfrak{M}$ is invariant by $Ad\,K_0$. As in the proof of Lemma for Proposition 2.2, let τ be the involutive automorphism of $\mathfrak{G}$ defined by $\tau|_{\mathfrak{K}} =$ identity and $\tau|_{\mathfrak{P}} = -$identity. Since $\tau \circ (ad\,Z) = -(ad\,Z) \circ \tau$ as we have shown earlier in the proof of Proposition 2.2, we have $\tau \circ (ad\,Z)^2 = (ad\,Z)^2 \circ \tau$. Hence $(ad\,Z)^2$ leaves $\mathfrak{K}$ and $\mathfrak{P}$ invariant. Since $ad\,Z$ leaves the Killing form (,) invariant, $(ad\,Z)^2$ is a symmetric endomorphism of $\mathfrak{G}$ with respect to (,). If we denote by $\mathfrak{P}_+$ the direct sum of the eigen-spaces corresponding to the positive eigen-values of $(ad\,Z)^2|_{\mathfrak{P}}$, then $\mathfrak{P} = \mathfrak{P}_0 + \mathfrak{P}_+$, and $\mathfrak{P}_0$ and $\mathfrak{P}_+$ are mutually orthogonal with respect to the Killing form (,). Since $(ad\,Z)^2$ maps $\mathfrak{K}_0$ into 0, $(ad\,Z)^2$ leaves $\mathfrak{M}$ invariant. Let $\gamma_1, \cdots, \gamma_n$ be the set of roots γ (multiplicity counted) of $\mathfrak{G}$ with respect to $\mathfrak{A}$ such that $\gamma(Z) > 0$. Then we know (Takeuchi [16]) that there exist a basis $S_1, \cdots, S_n$ for $\mathfrak{M}$ and a basis $T_1, \cdots, T_n$ for $\mathfrak{P}_+$ such that

$$(*) \quad \begin{aligned} &-(S_i, S_j) = \delta_{ij}\,, \quad (T_i, T_j) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n; \\ &[H, S_i] = \gamma_i(H)T_i\,, \ [H, T_i] = \gamma_i(H)S_i \quad \text{for } H \in \mathfrak{A} \text{ and } 1 \leq i \leq n; \\ &S_i + T_i \in \mathfrak{U} \qquad \text{for} \quad 1 \leq i \leq n. \end{aligned}$$

By setting $H = Z$ in $(*)$, we see that $[Z, \mathfrak{M}] = \mathfrak{P}_+$ and $[Z, \mathfrak{P}_+] = \mathfrak{M}$ and that $(ad\,Z)^2|_{\mathfrak{M}}$ is a positive definite symmetric endomorphism of $\mathfrak{M}$ with respect to $-(,)$. Let ζ be a positive definite symmetric endomorphism of $\mathfrak{M}$ with respect to $-(,)$ such that $\zeta^2 = (ad\,Z)^2|_{\mathfrak{M}}$. Then $\zeta S_i = \gamma_i(Z)S_i$ for $1 \leq i \leq n$. Since $(Ad\,k)Z = Z$ for $k \in K_0$, we have $(Ad\,k)\zeta X = \zeta(Ad\,k)X$ for $X \in \mathfrak{M}$ and $k \in K_0$.

Lemma 1. *$X + \zeta^{-1}[Z, X] \in \mathfrak{U}$ for $X \in \mathfrak{P}_+$.*

Proof of Lemma 1. It suffices to verify for $X = T_i$ $(1 \leq i \leq n)$. From $(*)$ we obtain

$$T_i + \zeta^{-1}[Z, T_i] = T_i + \zeta^{-1}\gamma_i(Z)S_i = T_i + \zeta^{-1}\zeta S_i = T_i + S_i \in \mathfrak{U}\,,$$

which proves Lemma 1.

We shall now construct K-invariant Riemannian metric $\ll\ ,\ \gg$ on $M = K/K_0$. Let $T_O(M)$ be the tangent space of $M = K/K_0$ at the origin o. Under the natural identification of $\mathfrak{M}$ with $T_O(M)$, the adjoint action of K_0 on $\mathfrak{M}$ corresponds to the linear isotropy representation of K_0 on $T_O(M)$. We set

$$\ll X, Y \gg \; = -(\zeta X, Y) \quad \text{for} \quad X, Y \in \mathfrak{M} .$$

Since $(\ ,\ )$ is negative definite on $\mathfrak{K}$ and ζ commutes with $Ad\,k$ on $\mathfrak{M}$ for every $k \in K_0$, it follows that $\ll\ ,\ \gg$ is a K_0-invariant positive definite symmetric bilinear form on $\mathfrak{M}$. Hence $\ll\ ,\ \gg$ can be extended uniquely to a K-invariant Riemannian metric $\ll\ ,\ \gg$ on $M = K/K_0$.

Let $X \in \mathfrak{P}$ and let u_X denote the linear functional on $\mathfrak{P}$ defined by $u_X(Y) = (Y, X)$ for $Y \in \mathfrak{P}$. Let φ be the imbedding of M into $\mathfrak{P}$ defined in § 2, and set $f_X = u_X \circ \varphi$. In other words, f_X is defined by

$$f_X(ko) = ((Ad\,k)Z, X) \quad \text{for} \quad k \in K .$$

Lemma 2. *For every $X \in \mathfrak{P}$, df_X is the 1-form (i.e., the covariant vector) corresponding to the vector field (i.e., the contravariant vector) generated by X under the duality defined by the Riemnnian metric $\ll\ ,\ \gg$.*

Proof of Lemma 2. We denote by the same letter X the vector field on M generated by X. The value of X at a point ko of M will be denoted by Xko. Similarly, for $Y \in \mathfrak{M}$, kYo denotes the vector at ko obtained from the vector $Yo \in T_O(M)$ by a transformation $k \in K$. Then Lemma 2 may be stated as follows:

$$<(df_X)_{ko}, kYo> \; = \; \ll Xko, kYo \gg \quad \text{for} \quad Y \in \mathfrak{M} \quad \text{and} \quad k \in K .$$

We calculate the left hand side first.

$$\begin{aligned} <(\mathrm{d}f_X)_{ko}, kYo> &= \frac{d}{dt} f_X((k \cdot \exp tY)o)|_0 = \frac{d}{dt}((Ad\,k \cdot \exp tY)Z, X)|_0 \\ &= \frac{d}{dt}((Ad \exp tY)Z, (Ad\,k^{-1})X|_0 = ([Y, Z], (Ad\,k^{-1})X) \\ &= (Y, [Z, (Ad\,k^{-1})X]) . \end{aligned}$$

We decompose $(Ad\,k^{-1})X \in \mathfrak{P}$ as follows: $(Ad\,k^{-1})X = X_0 + X_+$, where $X_0 \in \mathfrak{P}_0$ and $X_+ \in \mathfrak{P}_+$. Then we have

$$<(df_X)_{ko}, kYo> \; = (Y, [Z, X_+]) .$$

We now calculate the right hand side.

$$\ll Xko, kYo \gg \; = \; \ll ((Ad\,k^{-1})X)o, Yo \gg .$$

Since we have $((Ad\, k^{-1})X)o = (-\zeta^{-1}[Z, X_+])o$ by Lemma 1, we obtain

$$\langle\!\langle Xko, kYo \rangle\!\rangle = -\langle\!\langle \zeta^{-1}[Z, X_+], Y \rangle\!\rangle = ([Z, X_+], Y) .$$

This completes the proof of Lemma 2.

Theorem 3.1 now follows from Propositions 2.1 and 2.2 and from Lemma 2 just proved.

Remark 1. Given an R-space $M = G/U$ we may assume without loss of generality that G acts effectively on M, i.e., U contains no nontrivial normal subgroup of G. Then the minimum imbedding $\varphi : M \to \mathfrak{P}$ is substantial in the sense that $\varphi(M)$ is not contained in any (affine) hyperplane of $\mathfrak{P}$; otherwise there would exist a nonzero linear functional u_X of $\mathfrak{P}$ such that the function $f_X = u_X \circ \varphi$ is constant on M. But Lemma 2 says that if $df_X = 0$ on M, then the vector field on M generated by X also vanishes identically on M. Hence, $X = 0$.

Remark 2. Since $\beta \geq \sum \dim H_i(M, Z_p)$ by Morse theory, we may conclude that, for any R-space $M = G/U$, the inequality

$$\sum \dim H_i(M, Z_2) \geq \sum \dim H_i(M, Z_p)$$

holds for all prime numbers p.

4. Symmetric R-spaces and minimal submanifolds of spheres

Let G be a connected real semi-simple Lie group without center, and Z an element of $\mathfrak{G}$ such that $ad\,Z$ is a semi-simple endomorphism of $\mathfrak{G}$ with eigen-values -1, 0 and 1. Let $\mathfrak{G} = \mathfrak{G}_{-1} + \mathfrak{G}_0 + \mathfrak{G}_1$ be the corresponding eigen-space decomposition, and U the parabolic subgroup of G with Lie algebra $\mathfrak{U} = \mathfrak{G}_0 + \mathfrak{G}_1$. Taking a Cartan decomposition $\mathfrak{G} = \mathfrak{K} + \mathfrak{P}$ such that $Z \in \mathfrak{P}$, let K be the maximal compact subgroup of G generated by $\mathfrak{K}$. Let $K_0 = \{k \in K; (ad\,k)Z = Z\}$ and $\mathfrak{K} = \mathfrak{K}_0 + \mathfrak{M}$ as in §§2 and 3. Let $\mathfrak{G}_C$ be the complexification of $\mathfrak{G}$ and G_C the complex semi-simple Lie group without center generated by $\mathfrak{G}_C$. Let θ denote the restriction to K of the inner automorphism of $\mathfrak{G}_C$ defined by $\exp(\pi i Z) \in G_C$. If we set $K_\theta = \{k \in K; \theta k = k\}$, then K_0 lies between K_θ and the identity component of K_θ. It follows that $M = K/K_0$ is a symmetric space defined by the involutive automorphism θ of K. (By results of Nagano [13] (cf. also Kobayashi-Nagano [8] and Takeuchi [16]), the converse is also true; namely, if $M = G/U$ is an R-space such that $M = K/K_0$ is symmetric, then U is determined by an element $Z \in \mathfrak{G}$ such that $ad\,Z$ has eigen-values $-1, 0, 1$.) Throughout this section we shall consider a symmetric R-space $M = G/U = K/K_0$, where U is determined by such a $Z \in \mathfrak{G}$. The main purpose of this section is to prove that, with respect to the imbedding $\varphi : M \to \mathfrak{P}$ defined in §2, $\varphi(M)$ is a minimal submanifold of the sphere of radius $\sqrt{2n}$ in $\mathfrak{P}$, where $n = \dim M$.

With our notations in § 3, we have $\gamma_i(Z) = 1$ for $1 \leq i \leq n$ and $\zeta(X) = X$ for all $X \in \mathfrak{M}$. The Riemannian metric $\langle\!\langle\ ,\ \rangle\!\rangle$ on M is defined by $\langle\!\langle X, Y \rangle\!\rangle = -(X, Y)$ for $X, Y \in \mathfrak{M} = T_O(M)$. From the formulas $(*)$ in § 3 it follows that the imbedding $\varphi: M \to \mathfrak{P}$ is isometric with respect to the Riemannian metric $\langle\!\langle\ ,\ \rangle\!\rangle$ and the restriction of the Killing form $(\ ,\ )$ of $\mathfrak{G}$ to $\mathfrak{P}$.

From the definition of the imbedding $\varphi: M \to \mathfrak{P}$ it is clear that its image $\varphi(M)$ lies on the sphere of radius $(Z, Z)^{\frac{1}{2}}$ with center at the origin of $\mathfrak{P}$.

Proposition 4.1. *For a symmetric R-space $M = G/U$, we have $(Z, Z) = 2n$, where $n = \dim M$.*

Proof. $(Z, Z) = \operatorname{Tr}(ad\, Z)^2 = \sum_{i=1}^{n} \gamma_i(Z)^2 + \sum_{i=1}^{n} (-\gamma_i(Z))^2 = 2n.$

Theorem 4.2. *Let $M = G/U = K/K_0$ be a symmetric R-space with G simple. Then $\varphi(M)$ is a minimal submanifold of the sphere of radius $\sqrt{2n}$ about the origin in $\mathfrak{P}$, where $n = \dim M$.*

Proof. We identify $\varphi(M)$ with M. Let S denote the sphere of radius $\sqrt{2n}$ about the origin in $\mathfrak{P}$, and α be the second fundamental form of M in S; at each point $x \in M$, it defines a symmetric bilinear mapping $T_x(M) \times T_x(M) \to T_x^{\perp}$, where $T_x^{\perp}$ denotes the normal space to M in S at x. Choosing an orthonormal basis $e_1, \cdots, e_n$ for $T_x(M)$, we define the mean curvature normal ξ_x by

$$\xi_x = \sum_{i=1}^{n} \alpha(e_i, e_i)\,.$$

Then ξ_x is independent of the choice of $e_1, \cdots, e_n$. The submanifold M is minimal if and only if $\xi_x = 0$ at every point x of M. In the present case, since the imbedding φ is K-equivariant, the field ξ of mean curvature normals is invariant by the adjoint action of K in $\mathfrak{P}$. It suffices therefore to prove that ξ vanishes at the origin o of M. The tangent space $T_O(M)$ is parallel to $[Z, \mathfrak{M}] = \mathfrak{P}_+$ in $\mathfrak{P}$ (cf. formulas $(*)$ in § 3). Since Z is normal to the sphere S at o, ξ_O is perpendicular to Z as well as to $\mathfrak{P}_+$. Hence ξ_O can be identified with an element of $\mathfrak{P}_0$ which is perpendicular to Z and is invariant by the adjoint action of K_0 in $\mathfrak{P}_0$. The proof of the theorem is now reduced to that of the following lemma.

Lemma. *Let $M = G/U$ be a symmetric R-space with G simple. Then the space $\{X \in \mathfrak{P}_0;\ (Ad\, k)X = X \text{ for all } k \in K_0\}$ is spanned by Z.*

Proof of Lemma. Consider first the case where the complexification $\mathfrak{G}_C$ of $\mathfrak{G}$ is not simple. In this case, $\mathfrak{K}$ is compact and simple, and $\mathfrak{G}$ admits a complex structure J such that $\mathfrak{P} = J\mathfrak{K}$ and $\mathfrak{P}_0 = J\mathfrak{K}_0$. Moreover, $\mathfrak{K}_0$ has center of dimension 1 (cf. Helgason [3]). Our lemma is clearly true in this case.

Consider now the case where $\mathfrak{G}_C$ is simple. In this case, the center of $\mathfrak{G}_0$ is spanned by Z (cf. Kobayashi-Nagano [8] and Takeuchi [16]). Let $\mathfrak{G}_0' = [\mathfrak{G}_0, \mathfrak{G}_0]$ and $\mathfrak{P}_0' = \mathfrak{G}_0' \cap \mathfrak{P}_0$. Then $\mathfrak{G}_0' = \mathfrak{K}_0 + \mathfrak{P}_0'$ is a Cartan decomposition of a semi-simple Lie algebra $\mathfrak{G}_0'$. It follows that no nonzero element of $\mathfrak{P}_0'$ is invariant by $\mathfrak{K}_0$ (cf. Helgason [3]). Since the center of $\mathfrak{G}_0$ is spanned by Z, we have $\mathfrak{P}_0 = \mathfrak{P}_0' + \{Z\}_{\boldsymbol{R}}$.

Remark. The lemma above may be derived also from Frobenius reciprocity and the theorem of E. Cartan to the effect that every complex irreducible representation of K appears with multiplicity at most 1 in the regular representation of K on K/K_0.

5. Eigen-values of the Laplacian

Let $\boldsymbol{R}^{N+1}$ be a Euclidean space of dimension $N+1$ with natural coordinate system $y = (y^1, \cdots, y^{N+1})$. Let $S^N(r)$ be the sphere of radius r about the origin of $\boldsymbol{R}^{N+1}$, M an n-dimensional submanifold of $S^N(r)$ with local coordinate system $x^1, \cdots, x^n$, and

$$y = y(x^1, \cdots, x^n)$$

the local equation defining M. At each point of M, we choose an orthonormal system of unit vectors $\xi_0, \xi_1, \cdots, \xi_{N-n}$ such that ξ_0 is normal to $S^N(r)$ and $\xi_1, \cdots, \xi_{N-n}$ are tangent to $S^N(r)$ but normal to M. Then

$$\frac{\partial^2 y}{\partial x^j \partial x^k} = \sum_i \Gamma^i_{jk} \frac{\partial y}{\partial x^i} + \sum_{\lambda=1}^{N-n} b^\lambda_{jk}\xi_\lambda + b^0_{jk}\xi_0 .$$

If we set $g_{jk} = \left(\frac{\partial y}{\partial x^j}, \frac{\partial y}{\partial x^k}\right)$ and denote by (g^{jk}) the inverse matrix of (g_{jk}), then the Laplacian of $y = (y^1, \cdots, y^{N+1})$ as a system of functions on M is given by

$$\Delta y = \sum_{j,k} g^{jk}\nabla_j\nabla_k y = \sum_{\lambda,j,k} g^{jk}b^\lambda_{jk}\xi_\lambda + \sum_{j,k} g^{jk}b^0_{jk}\xi_0 ,$$

where ∇_j denotes the covariant differentiation with respect to $\partial/\partial x^j$. The first term on the right hand side is nothing but the so-called mean curvature normal on M as a submanifold of $S^N(r)$. Hence, M is a minimal submanifold of $S^N(r)$ if and only if

$$\Delta y = \sum_{j,k} g^{jk}b^0_{jk}\xi_0 .$$

To simplify the right hand side, we note that

$$(y, y) = r^2 , \quad \left(\frac{\partial y}{\partial x^j}, y\right) = 0 ,$$

$$\left(\frac{\partial^2 y}{\partial x^j \partial x^k}, y\right) + \left(\frac{\partial y}{\partial x^j}, \frac{\partial y}{\partial x^k}\right) = 0 .$$

Since $y = r\xi_0$ on M, the last equality above may be rewritten as follows:

$$rb^0_{jk} + g_{jk} = 0 .$$

Hence, $\sum_{j,k} g^{jk}b^0_{jk}\xi_0 = -\frac{n}{r^2}$ y. We may now conclude

Proposition 5.1. *A submanifold M of $S^N(r)$ is a minimal submanifold of $S^N(r)$ if and only if*

$$\Delta y = -\frac{n}{r^2} y ,$$

where $n = \dim M$.

From Theorem 4.2 and Proposition 5.1 we obtain

Theorem 5.2. *Let $M = G/U = K/K_0$ be a symmetric R-space with G simple, and $\varphi : M \to \mathfrak{P}$ the imbedding defined in § 2. For each linear functional u of $\mathfrak{P}$, we set $f = u \circ \varphi$. Then with respect to the metric $\langle\langle\ ,\ \rangle\rangle$ on M, f satisfies $\Delta f = -\frac{1}{2}f$.*

Remark. The fact that $\Delta f = \lambda f$ for some λ (independent of f) may be derived from the theorem of Cartan quoted in the remark at the end of § 4. We can then verify $\lambda = -1/2$ using the special function $f_Z = u \circ \varphi$.

We wish to relate this eigen-value $-1/2$ with the scalar curvature of M. We denote by $(\ ,\ )_{\mathfrak{G}}$ and $(\ ,\ )_{\mathfrak{K}}$ the Killing forms of $\mathfrak{G}$ and $\mathfrak{K}$, respectively. The curvature tensor R of the symmetric space $M = K/K_0$ is given by

$$R(V, X)Y = -[[V, X], Y] \quad \text{for} \quad V, X, Y \in \mathfrak{M};$$

its Ricci tensor S is given by

$$\begin{aligned} S(X, Y) &= \text{trace of the map } V \to R(V, X)Y \\ &= \text{trace of the map } V \to -[[V, X], Y] . \\ &= -\text{trace } ((ad\, Y)(ad\, X))|_{\mathfrak{M}} . \end{aligned}$$

If we construct an orthonormal basis for $\mathfrak{K}$ with respect to $-(\ ,\ )_{\mathfrak{G}}$ by choosing first an orthonormal basis for $\mathfrak{K}_0$ and then one for $\mathfrak{M}$, $ad\, X$ acting on $\mathfrak{K}$ is given by a matrix of the form

$$\begin{pmatrix} 0 & A(X) \\ -{}^tA(X) & 0 \end{pmatrix} .$$

Hence, $(ad\, Y)(ad\, X)$ acting on $\mathfrak{K}$ is given by a matrix of the form

$$\begin{pmatrix} -A(Y){}^tA(X) & 0 \\ 0 & -{}^tA(Y)A(X) \end{pmatrix} .$$

It follows that

$$(X, Y)_{\mathfrak{K}} = \text{trace}\,(ad\,Y)(ad\,X)|_{\mathfrak{K}} = -2(\text{trace}\,{}^tA(Y)A(X)) \\ = 2\,\text{trace}\,(ad\,Y)(ad\,X)|_{\mathfrak{M}} = -2S(X, Y)\,.$$

Proposition 5.3. *The Ricci tensor S of a symmetric space $M = K/K_0$ is given by*

$$S(X, Y) = -\tfrac{1}{2}(X, Y)_{\mathfrak{K}} \quad \textit{for} \quad X, Y \in \mathfrak{M}\,.$$

It we multiply the metric tensor of M by a positive constant a, then both the scalar curvature c of M and the Laplacian Δ of M are multiplied by $1/a$. It is therefore desirable to express the eigen-values of Δ in terms of c. Now we calculate c for some R-spaces. If there exists a positive number μ such that

$$(X, Y)_{\mathfrak{K}} = \mu\cdot(X, Y)_{\mathfrak{S}} \quad \text{for} \quad X, Y \in \mathfrak{K}\,,$$

then the scalar curvature c is given by

$$c = \tfrac{1}{2}n\mu \qquad (n = \dim M)\,.$$

In fact, for $X, Y \in \mathfrak{M}$, we have

$$S(X, Y) = -\frac{1}{2}(X, Y)_{\mathfrak{K}} = -\frac{\mu}{2}(X, Y)_{\mathfrak{S}} = -\frac{\mu}{2}\langle\langle X, Y\rangle\rangle\,,$$

and hence $c = \frac{1}{2}n\mu$. For the following six classes of symmetric spaces, this method enables us to calculate the scalar curvature c. (For calculation of μ, we refer the reader to Iwahori [5].)

(1) Irreducible hermitian symmetric space of compact type:

$$\mu = \frac{1}{2}\,, \qquad c = \frac{n}{4}\,.$$

(2) Real Grassmann manifold of non-oriented p-planes in $\boldsymbol{R}^{p+q}$, $(p + q > 2)$:

$$\mu = \frac{p + q - 2}{2(p + q)}\,, \qquad c = \frac{pq(p + q - 2)}{4(p + q)}\,.$$

(3) Quaternionic Grassmann manifold of p-planes in quaternionic vector space of dimension $p + q$:

$$\mu = \frac{p + q + 1}{2(p + q)}\,, \qquad c = \frac{pq(p + q + 1)}{p + q}\,.$$

(4) Group manifold $SO(m)$, $(m > 2)$:

$$\mu = \frac{m-2}{2m-2}, \qquad c = \frac{1}{8} m(m-2).$$

(5) Group manifold $Sp(m)$:

$$\mu = \frac{m+1}{2m+1}, \qquad c = \frac{1}{2} m(m+1).$$

(6) n-sphere, $(n > 1)$:

$$\mu = \frac{n-1}{n}, \qquad c = \frac{1}{2}(n-1).$$

By calculating the eigen-values of the Casimir operator, Nagano [12] determined the eigen-values of the Laplacian Δ acting on the space of functions on a compact symmetric space K/K_0 with K simple and K/K_0 simply connected (with respect to the invariant Riemannian metric induced from the Killing form of $\mathfrak{K}$). From Nagano's table we see that, for (1), (3) and (6), there is no eigen-value of Δ between 0 and $-\frac{1}{2}(=-c/(n\mu))$. Every eigen-value of Δ for functions on the Grassmann manifold of non-oriented p-planes in $\boldsymbol{R}^{p+q}$ appears as an eigen-value of Δ for functions on the Grassmann manifold of oriented p-planes in $\boldsymbol{R}^{p+q}$, but not vice versa. From Nagano's table we see that the Laplacian Δ for functions on the Grassmann manifold of non-oriented p-planes in $\boldsymbol{R}^{p+q}$ has no eigen-value between 0 and $-\frac{1}{2}\left(= -\frac{2c(p+q)}{pq(p+q-2)}\right)$ at least if $p \geq 3$ and $p+q \geq 17$. But we do not know if this is true for all p and q. By the same method we can verify that the Laplacian acting on the space of functions on the group manifold $SO(m)$ (resp. $Sp(m)$) has no eigen-value between 0 and $-\frac{1}{2}\left(= -\frac{4c}{m(m-2)}\right)$ $\left(\text{resp. } 0 \text{ and } -\frac{1}{2}\left(= -\frac{c}{m(m+1)}\right)\right)$. For eigen-values of the Laplacian for the spaces (1) and (6), see also Obata [14].

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